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Preprint No 57/2003

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COUPLING OF THE LEGENDRE POLYNOMIALS WITH KERNELS $|x-y|^\alpha$ and $\ln|x-y|$

Moscow, 2003

Abstract

Double integrals that represent matrix elements of the power and logarithmic potentials (resp. $|x-y|^\alpha$ and $\ln|x-y|$) in the Legendre polynomial basis are found in a closed form. Several proofs are given, which involve different special functions and identities. In particular, a connection of the new formulae and Whipple's hypergeometric summation formula is shown.

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§ 1. Introduction

In this paper we evaluate the integrals

$$B_\alpha(m,n) = \int_{-1}^{1}\int_{-1}^{1} |x-y|^\alpha\, P_m(x)\, P_n(y)\, dx\, dy \qquad (\operatorname{Re}\alpha > -1) \tag{1.1}$$

(matrix elements of the Bessel potentials) and

$$B'(m,n) = \frac{d}{d\alpha}\bigg|_{\alpha=0} B_\alpha(m,n) = \int_{-1}^{1}\int_{-1}^{1} \ln|x-y|\, P_m(x)\, P_n(y)\, dx\, dy\,. \tag{1.2}$$

Here P_n are the Legendre polynomials. Notation and relevant facts regarding orthogonal polynomials and hypergeometric series are collected in Appendix.

Sometimes it is more convenient to deal with the shifted Legendre polynoimials (A.6). Define

$$\tilde B_\alpha(m,n) = \int_{0}^{1}\int_{0}^{1} |x-y|^\alpha\, p_m(x)\, p_n(y)\, dx\, dy \qquad (\operatorname{Re}\alpha > -1) \tag{1.3}$$

and

$$\tilde B'(m,n) = \int_{0}^{1}\int_{0}^{1} \ln|x-y|\, p_m(x)\, p_n(y)\, dx\, dy. \tag{1.4}$$

It is readily seen that

$$\begin{aligned} B_\alpha(m,n) &= 2^{\alpha+2}\, \tilde B_\alpha(m,n), \\ B'(m,n) &= 4\left(\tilde B'(m,n) + \delta_{m0}\,\delta_{n0}\, \ln 2\right). \end{aligned}$$

Because of the symmetry of the integrands — see Appendix A.1.3. — all the above integrals vanish when $m-n$ is odd. In fact, we obtain more detailed results, which are non-trivial in the case of odd $m-n$ as well, — evaluation of the 'halves' of the above integrals. Instead of the interals over the square

$$-1 \le x \le 1, \quad -1 \le y \le 1 \qquad (\text{resp. } 0 \le x \le 1, \quad 0 \le y \le 1),$$

consider the integrals of the same integrands over the triangle

$$0 \le y \le x \le 1 \qquad (\text{resp. } 0 \le y \le x \le 1),$$

and denote them by $L_\alpha(m,n)$, $L'(m,n)$, $\tilde{L}_\alpha(m,n)$, $\tilde{L}'(m,n)$ respectively. Then

$$\begin{aligned} L(m,n) &= (-1)^{m-n}\, L(n,m)\,; \\ B(m,n) &= 2\, L(m,n) \quad \text{if } m-n \text{ is even,} \end{aligned} \tag{1.5}$$

with L being any of L_α, L', $\tilde{L}_\alpha$, $\tilde{L}'$ and the corresponding B. Yet we formulate theorems on evaluation of the B's separately, because there are compact proofs that do not involve evaluation of the L's.

In view of the symmetry (1.5), it is enough to bring the results assuming $m \ge n$. Denote $d = m-n$. Set

$$R_\alpha(m,n) \;=\; \frac{-1}{\alpha+1}\, \frac{((d-\alpha)/2)_n\,(-\alpha-1)_d}{((\alpha+d+4)/2)_n\,(\alpha+2)_{d+1}} \tag{1.6}$$

and

$$R'(m,n) \;=\; \frac{1}{(m+n)(m+n+2)(d^2-1)}. \tag{1.7}$$

Notice that if $d > 0$ then the fraction $R_\alpha(m,n)$ can be reduced by the common factor $(\alpha+1)$.

Theorem 1. *If $m-n \ge 0$ is even, then*

$$\tilde{B}_\alpha(m,n) \;=\; 2\, R_\alpha(m,n). \tag{1.8}$$

Theorem 2. (i) *If $m-n$ is even and $(m,n) \ne (0,0)$, then*

$$\tilde{B}'(m,n) \;=\; 2\, R'(m,n). \tag{1.9}$$

(ii) *Special case $m=n=0$:*

$$\tilde{B}'(0,0) \;=\; -3/2. \tag{1.10}$$

Correspondingly, for even nonnegative $d = m-n$,

$$B_\alpha(m,n) \;=\; 2^{\alpha+3}\, R_\alpha(m,n), \tag{1.11}$$

In particular,

$$B_\alpha(0,0) = \frac{2^{\alpha+3}}{(\alpha+1)(\alpha+2)}.$$

Also, assuming $m=n$ even and $(m,\, n) \ne (0,0)$, we have

$$B'(m,n) \;=\; 8\, R'(m,n). \tag{1.12}$$

The special case is

$$B'(0,0) = 4\ln 2 - 6 \;=\; -3.2274\ldots\,. \tag{1.13}$$

Theorem 3. *If* $m - n \geq 0$ *then*

$$\tilde{L}_\alpha(m,n) \;=\; R_\alpha(m,n). \tag{1.14}$$

Theorem 4. (i) *If* $m - n \geq 2$ *or* $m = n \neq 0$, *then*

$$\tilde{L}'(m,n) \;=\; R'(m,n). \tag{1.15}$$

(ii) *Special case* $m - n = 1$ *:*

$$\tilde{L}'(n+1,n) \;=\; -\frac{1}{q}\left(H^{\rm odd}_{2n+1} - \frac{1}{4} - \frac{1}{q}\right), \tag{1.16}$$

where

$$q \;=\; (2n+1)(2n+3) \;=\; (m+n)(m+n+2),$$

and $H^{\rm odd}_{2n+1}$ *is the* $n+1$*-th odd harmonic sum,*

$$H^{\rm odd}_{2n+1} = 1 + \frac{1}{3} + \ldots + \frac{1}{2n+1}. \tag{1.17}$$

The following numerical table supports Theorem 4 and includes the omitted case $m = n = 0$. Index m enumerates the rows and n the columns.

$$\left[\tilde{L}'(m,n)\right]_{m,n=0\ldots3} \;=\; \begin{bmatrix} \frac{-3}{4} & \frac{5}{36} & \frac{1}{24} & \frac{1}{120} \\ \frac{-5}{36} & \frac{-1}{8} & \frac{61}{900} & \frac{1}{72} \\ \frac{1}{24} & \frac{-61}{900} & \frac{-1}{24} & \frac{527}{14700} \\ \frac{-1}{120} & \frac{1}{72} & \frac{-527}{14700} & \frac{1}{120} \end{bmatrix}$$

The paper is organized as follows.

Section 2 contains a one-page proof of Theorem 2, due to M. Rahman [11].

Section 3 contains a short proof of Theorem 1 proposed by R. Askey. It involves three interesting formulae from the theory of special functions. The proof is simple because nasty details are worked out in the cited results.

In **Section 4** we prove Theorem 3 by transforming the integral $\tilde{L}_\alpha(m,n)$ to a certain terminating hypergeometric series, which can be evaluated by a formula due to Whipple. Moreover, through this connection we prove the terminating case of Whipple's formula. (Essense: our integral is a rational function with known zeros and poles.) A proof of Whipple's formula in the general case is then obtained using a standard function-theoretic argument.

Finally, in **Section 5** we give an independent, self-contained, and elementary proof of Theorem 4, which had been largely computer-aided.

This work stems from my study of the integral $B'(m,n)$ that was originally motivated by numerical solution of integral equations on polygons. Formula (1.12) was published without proof in [12], [13]. The following problem about a generalization of the integral $\tilde{B}'(m,n)$ is important in those applications.

Problem. *Propose an efficient numerical procedure for evaluation of the integral*

$$\tilde{B}'(m,n;\,\lambda) = \int_0^1\!\!\int_0^1 \ln(\lambda x - y)\, p_m(x)\, p_n(y)\, dx\, dy,$$

where λ *is a complex parameter. The method must be numerically stable and suitable for large values of* m *and* n*, and use the ordinary floating point arithmetics.*

The $\tilde{B}'(m,n;\,\lambda)$ are rational functions of λ, if $m-n\geq 2$. Their evaluation using computer algebra with exact rational arithmetics is not what we wish, since as $|m-n|$ grows, the expressions become huge and the calculation slow.

We conclude this section by deriving Theorem 2 from Theorem 1.

Case $n=d=0$ stands alone, but it is trivial. Cases (i) $d\neq 0$ and (ii) $d=0\,,n>0$ need separate treatment, but in both cases $\tilde{B}_\alpha$ is a rational function of α with a simple zero at $\alpha=0$. It remains to factor out the α and to evaluate the remaining quotient at $\alpha=0$. In case (i) the α is contained in $(-\alpha-1)_d$. We have: as $\alpha\to 0$

$$\frac{\tilde{B}_\alpha(m,n)}{\alpha} \;\to\; \frac{2\,(\frac{d}{2})_n\,(1)_{d-2}}{(\frac{d}{2}+2)_n\,(2)_{d+1}} = \frac{2\,\frac{d}{2}\,(\frac{d}{2}+1)}{(\frac{d}{2}+n+1)(\frac{d}{2}+n)(d-1)d(d+1)(d+2)}.$$

Simplifying, we obtain (1.9). In case (ii) the factor α comes from $(d-\alpha/2)_n=(-\alpha/2)_n$. When $\alpha\to 0$,

$$\alpha^{-1}\,\tilde{B}_\alpha(n,n) \;\to\; -(1)_{n-1}/(2\cdot(2)_n) \;=\; -1(2n(n+1))^{-1},$$

in agreement with (1.9).

Acknowledgements

Prof. *M. Rahman* (Ottawa) found the first proof of the formula (1.12) and communicated it to me on Feb. 15, 1999.

Profs. *G. Gasper* (Northwestern University), *R. Askey* (Wisconsin) and *E.D. Krupnikov* (Novosibirsk) responded to my question posted in an online forum on special functions (opsftalk@nist.gov) in February 2001. regarding formula (B.10); at that time I didn't know that it was Whipple's. Prof. Askey also provided a sketch of proof of Theorem 1 (see Sect. 3).

Publication of this work has been supported by the Russian Foundation for Basic Research under grants 02-01-01067 and 01-01-00517.

§ 2. M. Rahman's proof of Theorem 2

In this proof, the double integral is turned to a sum of two ordinary integrals of a Legendre polynomial and a Legendre function of the 2nd kind, for which the values are known. We work with polynomials P_n, rather than with the shifted Legendre polynomials, so the target is formula (1.12).

Lemma 2.1. *Denote*

$$F_n(x) = \frac{2n+1}{2} \int_{-1}^{1} P_n(y) \ln|x-y| \, dy. \tag{2.1}$$

Then

$$F_n(x) \;=\; \frac{1}{2}\, \text{P.V.} \int_{-1}^{1} \frac{P_{n+1}(y) - P_{n-1}(y)}{x-y} \, dy. \tag{2.2}$$

Proof. Replacing $\ln|x-y|$ in (2.1) by $\ln(x-y)$ doesnt't affect real part of the integral, whatever branch of the logarithm is taken. Move the integration path to the upper complex half-plane to obtain a regular integrand, and integrate by parts, using (A.5). The boundary terms vanish, since $[P_{n+1} - P_{n-1}](\pm 1) = 0$. Putting the integration path back on the real line, we obtain (2.2). □

The integral in (2.2) is the difference of two Legendre functions of the second kind, $Q_{n-1}(x)$ and $Q_n(x)$ [6], **3.6** (29). Therefore,

$$\frac{2n+1}{2}\, B'(m,n) = \int_{-1}^{1} P_m(x)\, F_n(x)\, dx = I(m,\, n-1) - I(m,\, n+1), \tag{2.3}$$

where

$$I(m,k) \;=\; \int_{-1}^{1} P_m(x)\, Q_k(x)\, dx.$$

The integrals $I(m,k)$ are evaluated in [6], **3.12** (17). The result is:

(i) if $m-k$ is even, then $I(m,k) = 0$;

(ii) if $m-k$ is odd, then

$$I(m,k) \;=\; \frac{2}{(m-k)\,(k+m+1)}. \tag{2.4}$$

Assuming that $m - n = d$ is even, substituion of (2.4) to (2.3) gives

$$\frac{2n+1}{2}\, B'(m,n) \;=\; \frac{2}{(d+1)\,(m+n)} \;-\; \frac{2}{(d-1)\,(m+n+2)},$$

and (1.12) easily follows. ■

§ 3. R. Askey's proof of Theorem 1

In response to my question regarding formula (1.11) posted in the online forum on special functions, R. Askey [3] wrote: [1]

There is a long paper by Polya and Szego in the early 1930s, see either Polya's collected papers or those of Szegö, in which they find an expansion of $|x-y|^\alpha$ *as a sum of products of ultraspherical polynomials. Then the integral (1.1) can be evaluated as a single hypergeometric series by using the formula for the integral of an ultraspherical polynomial times a Legendre polynomial. This is a special case of a formula of Gegenbauer which I have used frequently. There are a number of relatively simple derivations of this formula. One of the easiest is the one Gasper and I gave in the paper we wrote for the meeting celebrating de Branges's proof of the Bieberbach conjecture. It just uses the generating function of ultraspherical polynomials, differentiation, and orthogonality to get the result. Then it is just a matter of seeing if the hypergeometric series which comes from these calculations can be summed. Sadov claimed a specific formula is true, so the series can be summed.*

This section supplies details of Askey's approach.

Step 1. Decomposition. Ultraspherical (or Gegenbauer's) polynomials $P_n^{(\nu)}(x)$ are defined by the generating function

$$(1 - 2\,x\,t + t^2)^{-\nu} = \sum_{n=0}^{\infty} P_n^{(\nu)}(x)\, t^n. \tag{3.1}$$

The decomposition of $|x-y|^\alpha$ due to Polya and Szegö is [10, S. 27]

$$|x-y|^\alpha = M_1 \sum_{j=0}^{\infty} \left(1+\frac{j}{\nu}\right) P_j^{(\nu)}(x)\, P_j^{(\nu)}(y), \tag{3.2}$$

where $\nu = -\alpha/2$ and

$$M_1 = M_1(\alpha) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1+\nu)}{\Gamma(\frac{1}{2})}. \tag{3.3}$$

In (3.2), $\alpha > 0$. Convergence in the r.h.s. follows from the estimation

$$P_j^{(\nu)}(x) = O(j^{\nu-1}), \quad -1 \le x \le 1.$$

Step 2. Integration. The formula due to Gegenbauer reads [4]

$$\int_{-1}^{1} P_j^{(\nu)}(x)\, P_{j-2s}(x)\, dx = \frac{(\nu)_{j-s}\,(\nu-1/2)_s}{s!\,(1/2)_{j-s+1}}. \tag{3.4}$$

Assume $m-n=d \ge 0$ and even. Integral of the j-th term in (3.2) times $P_m(x)\,P_n(y)$ may be nonzero only when $j=m+2k$, $k \ge 0$. We have

$$\langle P_j^{(\nu)}(x),\, P_m(x)\rangle = \frac{(\nu)_{m+k}\,(\nu-1/2)_k}{k!\,(1/2)_{m+k+1}},$$

$$\langle P_j^{(\nu)}(y),\, P_n(y)\rangle = \frac{(\nu)_{n+k+d/2}\,(\nu-1/2)_{k+d/2}}{(k+d/2)!\,(1/2)_{n+k+d/2+1}}.$$

[1] Citation is slightly abridged.

Write the obtained sum in the standard hypergeometric form (B.5), with a prefactor, using the identity

$$1-\frac{2\,j}{\alpha}\;=\left(1+\frac{m}{\nu}\right)\,\frac{((m+\nu)/2+1)_k}{((m+\nu)/2)_k},$$

and the identities of the type (B.2), for example,

$$(\nu)_{n+k+d/2}\;=\;(\nu)_{n+d/2}\,(\nu+n+d/2)_k.$$

Also recall that $n+d/2=m-d/2$. The result of integration takes the form

$$B_\alpha(m,n)\;=\;M_1\,M_2\;\cdot\;{}_5F_4\left(\begin{matrix}\nu+m,\;1+\frac{m+\nu}{2},\;\nu-\frac{1}{2},\;\nu+m-\frac{d}{2},\;\nu+\frac{d-1}{2}\\ \frac{m+\nu}{2},\;\frac{3}{2}+m,\;\frac{d}{2}+1,\;\frac{3-d}{2}+m\end{matrix}\right) \tag{3.5}$$

with

$$M_2\;=\;\frac{(1+m/\nu)\,(\nu)_m\,(\nu)_{m-d/2}\,(\nu-1/2)_{d/2}}{(1/2)_{m+1}\,(1)_{d/2}\,(1/2)_{m-d/2+1}}. \tag{3.6}$$

Step 3. Summation. The above series ${}_5F_4$ can be summed by the limiting case (B.11) of Dougall's formula for a well-poised terminating ${}_7F_6$. The result is a ratio of products of Γ functions. Arguments of four Γ's in the numerator are

$$m+3/2,\quad d/2+1,\quad m+(3-d)/2,\quad \alpha+2. \tag{3.7}$$

In the denominator we have four Γ's with arguments

$$m+1-\alpha/2,\quad (3+\alpha)/2,\quad m+2+(\alpha-d)/2,\quad (3+\alpha+d)/2. \tag{3.8}$$

In total, we obtain an expression for $B_\alpha(m,n)$ in the form of a fraction with 10 factors in the numerator and 8 factors in the denominator. To simplify, we break down the triple product (3.5) into 5 groups and write

$$B_\alpha(m,n)\;=\;\Pi_1\,\Pi_2\,\Pi_3\,\Pi_4\,\Pi_5.$$

In the expressions for the Π's below, we show which little factors come from which big ones. The notation $[j,k]$ stands for the k-th term in the numerator or denominator of the j-th big factor: $j=1$ refers to M_1; $j=2$ to M_2; and $j=3$ to ${}_5F_4$. Denominator terms are shown in bold. For example, [2,1] is the first term in the numerator of (3.6), that is, $(1+m/\nu)$. One cancelation $[3,2]/[\mathbf{2},\mathbf{2}]=1$ occurs immediately and those terms are not attributed to any group. The grouping of other terms and the results of simplification are as follows

$$\begin{aligned}
\Pi_1\;&=\;[1,2]\;[2,1]\;[2,2]/[\mathbf{3},\mathbf{1}]\;=\;1,\\
\Pi_2\;&=\;[3,1]/[\mathbf{1},\mathbf{1}]\;[\mathbf{2},\mathbf{1}]\;=\;\Gamma(m+3/2),\\
\Pi_3\;&=\;[1,1]/[\mathbf{3},\mathbf{2}]\;=\;2\,/(\alpha+1),\\
\Pi_4\;&=\;[3,4]/[\mathbf{3},\mathbf{3}]\;[\mathbf{3},\mathbf{4}]\;=\;2^{2+\alpha+d}\,\pi^{-1/2}\;\Big[(\alpha+2)_{d+1}\,(2+\tfrac{\alpha+d}{2})_n\Big]^{-1}\\
\Pi_5\;&=\;\frac{[2,2]\;[2,3]\;[3,3]}{[\mathbf{2},\mathbf{3}]}\;=\;\frac{(-\alpha/2)_{m-d/2}\,(-(\alpha+1)/2)_{d/2}\,\Gamma(m+\frac{3-d}{2})}{(1/2)_{m-d/2+1}}.
\end{aligned}$$

The part $\Gamma(\ldots)/(1/2)_{m-d/2+1}$ in Π_5 cancels with $\pi^{-1/2}$ from Π_4. It remains to apply the duplication formula (B.4) with $a=-\alpha/2$, $n=d/2$, and (1.11) follows. ■

Limiting case: another proof of Theorem 2

A proof of Theorem 2, different from those given in Sections 2 and 5, can be obtained as a special limiting case of the above argument. It follows the same lines, but instead of the ultraspherical polynomials we encounter Chebyshev's polynomials T_n, where analogues of (3.2) and (3.4) are rather elementary. Chebyshev's polynomials show up here, because $\partial_\nu P_n^{(\nu)}(x)\Big|_{\nu=0} = (2/n)\,T_n(x)$ $(n>0)$ and the generating function for T_n's is

$$\ln(1 - 2\,x\,t + t^2) \;=\; -2\sum_{n=1}^{\infty} n^{-1}\,T_n(x)\,t^n.$$

The role of (3.2) is played by the formula

$$\ln|x-y| \;=\; -\ln 2 \;-\; 2\sum_{j=1}^{\infty} j^{-1}\,T_j(x)\,T_j(y),$$

which can be identified with the well known Fourier expansion

$$-\ln|2\,\sin(\theta/2)| \;=\; \sum_{j=1}^{\infty} |j|^{-1}\,\cos(j\theta)$$

using the definition of Chebyshev's polynomials $T_j(\cos\theta) = \cos(j\theta)$.
The analogue of (3.4) reads (for $j>0$)

$$\frac{2}{j}\int_{-1}^{1} T_j(x)\,P_{j-2s}(x)\,dx \;=\; -\frac{(s+1)_{j-2s-1}}{2\,(s-1/2)_{j-2s+2}}.$$

Step 3 again amounts to evaluation of the particular case of the series (3.6) with $\nu=0$ using Dougall's formula. (It seems attractive to find its elementary evaluation in this case.)

§ 4. Proof of Theorem 3 via Whipple's formula

4.1. Reduction of integral $\tilde{L}_\alpha(m,n)$ to a terminating ${}_3F_2$ series

Set

$$A_\alpha(n,\,x) = \int_0^x (x-y)^\alpha p_n(y)\,dy, \tag{4.1}$$

then

$$\tilde{L}_\alpha(m,n) \;=\; \int_0^1 p_m(x)\,A_\alpha(n,\,x)\,dx. \tag{4.2}$$

Lemma 4.1.

$$A_\alpha(n,\,x) \;=\; \frac{x^{\alpha+1}}{\alpha+1}\;{}_2F_1\left(\begin{matrix}-n, & n+1\\ & \alpha+2\end{matrix}\;;\,x\right). \tag{4.3}$$

In particular,

$$A_\alpha(n,\,1) \;=\; \frac{(-1)^n\,(-\alpha)_n}{(\alpha+1)_{n+1}}. \tag{4.4}$$

Proof. Apply the Rodrigues formula (A.7) and integrate by parts n times. Assume $\alpha > n$, then the boundary terms vanish and we get

$$A_\alpha(n,\,x) \;=\; \int_0^x \frac{(-\alpha)_n\,(-1)^n}{n!}\,(x-y)^{\alpha-n}\,y^n\,(1-y)^n\,dy.$$

Setting $y = tx$ leads to the integral (B.6) for ${}_2F_1$ with parameters $a = -n,\ b = n+1,$ $c = \alpha+2$, and the prefactor

$$x^{\alpha+1}\,\frac{(-1)^n\,(-\alpha)_n}{n!}\,\frac{\Gamma(n+1)\Gamma(\alpha-n+1)}{\Gamma(\alpha+2)},$$

which reduces to $x^{\alpha+1}/(\alpha+1)$. The assumption $\alpha > n$ can be dropped because both parts of (4.3) are analytic in α. □

Lemma 4.2.

$$\tilde{L}_\alpha(m,n) \;=\; \frac{(-\alpha-1)_m}{(\alpha+1)_{m+2}}\;{}_3F_2\left(\begin{matrix}-n,\;\; n+1,\;\; \alpha+2\\ \alpha+m+3,\;\; \alpha+2-m\end{matrix}\right). \tag{4.5}$$

Proof. Substituting (4.3) in (4.2), we find

$$\tilde{L}_\alpha(m,n) \;=\; \sum_{j=0}^{n}\frac{(-n)_j\,(n+1)_j}{j!\,(\alpha+1)_{j+1}}\int_0^1 x^{\alpha+1+j}\,p_m(x)\,dx.$$

Since $p_m(x) = (-1)^m\,p_m(1-x)$, the integral in the common term is

$$(-1)^m\,I(m,\,\alpha+1+j,\,1) \;=\; \frac{(-\alpha-1-j)_m}{(\alpha+2+j)_{m+1}},$$

according to (4.4). Using the identities (cf. (B.2))

$$(\alpha+1)_{j+1}\,(\alpha+j+2)_{m+1} \;=\; (\alpha+1)_{m+2}\,(\alpha+m+3)_j$$

and

$$(-\alpha-j-1)_m \;=\; (-\alpha-1)_m\frac{(\alpha+2)_j}{(\alpha-m+2)_j},$$

the common term is transformed into

$$\frac{(-\alpha-1)_m\,(-n)_j\,(n+1)_j\,(\alpha+2)_j}{(\alpha+1)_{m+2}\,(\alpha+m+3)_j\,(\alpha+2-m)_j},$$

so (4.5) follows. □

Remark. The symmetry (1.5) apparent in the l.h.s. of (4.5) is hidden in the r.h.s., where it is a particular case of Thomae's formula (B.8).

4.2. Equivalence of Theorem 1 and the terminating case of Whipple's formula

Invoke the terminating case of Whipple's summation formula (B.9) with

$$b = \alpha + m + 3, \qquad c = \alpha + 2 - m, \tag{4.6}$$

to evaluate the r.h.s. of (4.5), and get

$$\tilde{L}_\alpha(m,n) = 4^n \, \frac{(-\alpha-1)_m \, (\frac{\alpha+d+3}{2})_n \, (\frac{\alpha+2-m-n}{2})_n}{(\alpha+1)_{m+2} \, (\alpha+m+3)_n \, (\alpha+2-m)_n} .$$

To identify this expression with (1.6), decompose the ratio of the right sides of the two formulae into three groups and simplify using (B.2)–(B.4):

$$\frac{4^n \, ((\alpha+d+3)/2)_n \, ((\alpha+d+4)/2)_n}{(\alpha+1)_{m+2} \, (\alpha+m+3)_n \, (\alpha+d+3)_n} \;=\; \frac{1}{\alpha+1},$$

$$\frac{(-\alpha-1)_m}{(-\alpha-1)_d \, (\alpha+2-m)_n} \;=\; \frac{((\alpha+2-m-n)/2)_n}{((d-\alpha)/2)_n} \;=\; (-1)^n .$$

Conversely, (1.11) implies (B.9) in the case when the parameters b and c can be presented in form (4.6) with $m \in \mathbb{Z}_+$ and $\mathrm{Re}\, \alpha > -1$, that is if $b - c$ is an even non-negative integer and $\mathrm{Re}\,(b+c) > 4$. But both parts of (B.9) are rational functions of b and c. Hence the restrictions can be dropped. ■

4.3. Proof of Whipple's formula in the terminating case

The l.h.s. of (B.9) is a rational function of α of the form

$${}_3F_2\left(\begin{matrix} -n, \;\; n+1, \;\; \alpha+2 \\ \alpha+m+3, \;\; \alpha+2-m \end{matrix}\right) = \frac{Q(\alpha)}{(\alpha+m+3)_n(\alpha-m+2)_n}, \tag{4.7}$$

where $Q(\alpha)$ is a polynomial of degree $2n$ (of course Q depends on m and n). We shall find all roots of $Q(\alpha)$ by using relation (4.5) and studying zeros and poles of $\tilde{L}_\alpha(m,n)$. In this way we obtain a self-contained proof of Theorem 1 together with formula (B.9).

Lemma 4.3. *If $m - n > 0$ and even. Then* (i) $\tilde{L}_\alpha(m,n) = 0$ *for* $\alpha = 0, 2, \ldots, m+n-2$; (ii) $Q(m+n-2s) = 0$ *for* $s = 1, \ldots, n$.

Proof. (i) It is enough to show that $\tilde{B}_\alpha(m,n) = 0$ at the above values of α. In this case, $|x-y|^\alpha$ is a polynomial of degree $< m+n$. In its binomial expansion $c_{kl}\, x^k\, y^l$, the exponents k, l satisfy at least one of the inequalities $k < m$, $l < n$. Therefore, all products $\langle x^k,\, p_m(x)\rangle \, \langle y^l,\, p_n(y)\rangle$ are 0.

(ii) Using the relation

$$(-1-\alpha)_m/(\alpha+2-m)_n \;=\; (-1)^n \, (-\alpha-1)_{m-n},$$

we get

$$\tilde{L}_\alpha(m,n) \;=\; R(\alpha)\, Q(\alpha)\, \alpha\,(\alpha+1)\dots(\alpha+m-n-2),$$

where $R(\alpha)$ is a rational function, which does not have poles and zeros at nonnegative integers. Now the claim follows from (i). $\square$

In the next Lemma, we find n other roots of $Q(\alpha)$.

Lemma 4.4. *Let* $m>n$. *Then* (i) $\tilde{L}_\alpha(m,n)$ *does not have poles at* $\alpha=-m-n-1+2s$, $s=0,\dots,n-1$; (ii) $Q(-m-n-1+2s)=0$ *for* $s=0,\dots,n-1$.

Proof. (ii) follows from (i), because the values $\alpha=-m-n-1+2s,\ 0\le s\le n-1$, are uncompensated roots of the denominator in the r.h.s. in (4.5), while $\tilde{L}_\alpha(m,n)$ is regular at these values.
To prove (i), consider analytical continuation of $\tilde{L}_\alpha$. Using the identity

$$r^\alpha=(\Gamma(-\alpha))^{-1}\int_0^\infty e^{-rt}\,t^{-\alpha-1}\,dt \qquad (\mathrm{Re}\ \alpha<0),$$

we obtain for $-1<\mathrm{Re}\ \alpha<0$

$$\tilde{L}_\alpha(m,n)=\frac{1}{\Gamma(-\alpha)}\int_0^\infty t^{-\alpha-1}\int_0^1 e^{-tx}p_m(x)\int_0^x e^{ty}p_n(y)\,dy\,dx\,dt.$$

Divide the t-domain into $\int_0^1$ and $\int_1^\infty$; the first part is a regular function of α for $\mathrm{Re}\,\alpha<0$. Let us find asymptotics of the integrand as $t\to\infty$ in the second part and study its analytical continuation in $\mathrm{Re}\ \alpha\le -1$.

The following formula of indefinite integration

$$\int p(\xi)e^{\lambda\xi}\,d\xi=e^{\lambda\xi}(\lambda^{-1}p(\xi)-\lambda^{-2}p'(\xi)+\lambda^{-3}p''(\xi)-\dots) \tag{4.8}$$

holds for any polynomial $p(\xi)$. So we have

$$\int_0^x e^{ty}p_n(y)\,dy=e^{tx}\sum_{j=0}^n\frac{(-1)^j}{t^{j+1}}\,p_n^{(j)}(x)\;-\;\sum_{j=0}^n\frac{(-1)^j}{t^{j+1}}\,p_n^{(j)}(0).$$

Applying (4.8) once again and using orthogonality, we get

$$\int_0^1 e^{-tx}\,p_m(x)\int_0^x e^{ty}\,p_n(y)\,dy\,dx\;=\;\sum_{j=0}^n\sum_{k=0}^m\frac{(-1)^{j+1}}{t^{j+k+2}}\,p_n^{(j)}(0)\,p_m^{(k)}(0)\;+\;O(e^{-t}).$$

The coefficient at t^{-r} is the residual of $\tilde{L}_\alpha(m,n)$ at the pole $\alpha=-r$, due to

$$\int_1^\infty t^{-\alpha-1-r}\,dt=\frac{1}{\alpha+r}.$$

By (A.10), these coefficients vanish for $r=m+n+1-2s$, $0\le s\le n-1$. Hence there are no poles at the values stated in (i) of the Lemma. $\square$

The two Lemmas supply the complete list of $2n$ roots of the numerator $Q(x)$. Thus we obtain: *If $m-n$ is even and >0, then*

$$ {}_3F_2\left(\begin{matrix} -n, & n+1, & \alpha+2 \\ \alpha+m+3, & \alpha+2-m \end{matrix}\right) = C\,\frac{\left(\frac{\alpha+m+3-n}{2}\right)_n\left(\frac{\alpha-m+2-n}{2}\right)_n}{(\alpha+m+3)_n\,(\alpha-m+2)_n}\,, \tag{4.9} $$

where C is independent of α.

The left side is also a rational function of m with degrees $2n$ of the numerator and denominator, so the coefficient C is independent of m. Moreover, we can dismiss the above assumption of $m-n$ being even and positive and allow m to be non-integral. To determine C (which now may depend only on n), let us take $\alpha \to \infty$. Then the l.h.s. of (4.9) is $1+o(1)$, while the r.h.s. is $C/4^n + o(1)$. Thus $C=4^n$ and (B.9) follows by substitution (4.6). ∎

4.4. Proof of Whipple's formula in the general case

Fix the parameters b and c in Whipple's formula (B.10). The difference of the l.h.s. and the r.h.s. of (B.10) is an entire function $f(a)$. The motif of the proof is simple (cf. technique based on Carlson's theorem [5, Chapter V]). The function $f(a)$ has sufficiently many known zeros and a uniformly controlled growth as $|a|\to\infty$. From this we will conclude that $f(a)\equiv 0$.

At this time, the proof is not self-contained, though marginally. I have to recourse to additional facts from the theory of generalized hypergeometric series: Karlsson's formula and Dixon's formula.

Lemma 4.5. (i) *If* $\operatorname{Re}(b+c)>1$ *then the right-hand side of (B.10) has estimation* $O(e^{\pi|\operatorname{Im} a|}(1+|a|)^{1-\operatorname{Re}(b+c)})$ *as* $|a|\to\infty$.
(ii) *If b, c are real and $b+c>1$ then the left-hand side of (B.10) is* $O(e^{\pi|a|}(1+|a|)^{1-(b+c)})$ *as* $|a|\to\infty$.

Proof. (i) follows from the inequality

$$ (\Gamma(z)\Gamma(r-z))^{-1} \;<\; C(r)\,e^{\pi|\operatorname{Im} z|}\,(1+|z|)^{1-\operatorname{Re} r}, \tag{4.10} $$

which is a consequence of the formula $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ and the estimation $|\Gamma(x)/\Gamma(x+r)| = O(x^{-\operatorname{Re} r})$, $x\to\infty$, $|\arg x|<\pi$.

(ii) The series has the form

$$ \sum_{n=0}^{\infty} \mu_n\,\frac{(a)_n\,(1-a)_n}{n!\,((b+c+1)/2)_n}, $$

where

$$ \mu_n = \frac{((b+c-1)/2)_n\,((b+c+1)/2)_n}{(b)_n\,(c)_n} = O(1) \quad \text{as } n\to\infty. $$

Next, $(a)_n(1-a)_n$ is a series with positive coefficients in $t=(a-1/2)^2$. So maximum of our ${}_3F_2$ on the circle $|a-1/2|=R$ is bounded, up to a constant, by the respective maximum of

$$ {}_2F_1\left(\begin{matrix} a, & 1-a \\ (b+c+1)/2 \end{matrix}\right) \overset{(B.7)}{=} \frac{\Gamma(\frac{b+c+1}{2})\,\Gamma(\frac{b+c-1}{2})}{\Gamma(\frac{b+c+1}{2}-a)\,\Gamma(\frac{b+c-1}{2}+a)}. $$

Finally, by (4.10), the r.h.s. is $O(e^{\pi|\operatorname{Im} a|}\,(1+|a|)^{1-(b+c)})$. □

If in (ii) we had $e^{\pi|\operatorname{Im} a|}$ instead of $e^{\pi|a|}$, then the proof could be easily completed via the maximum principle applied to the function $f(a)\operatorname{cosec}\pi a$, which is entire. (No poles on $\mathbb{Z}$: the terminating case is already established).

In reality, we have only found that $\ln|f(a)| \le (\pi+o(1))|a|$. Consider the asymptotical density of zeros of f,

$$\nu = \varlimsup_{R\to\infty} N(R)/R, \qquad N(R) = \#\{\, a \;:\; f(a)=0 \text{ and } |a|<R \,\}$$

It follows from Jensen's formula that the estimation $\nu > 2$ would imply $f \equiv 0$. Thus far we only know that $f|\mathbb{Z} = 0$, which gives $\nu \ge 2$ but leaves a possibility of the equality. To complete the proof, we shall present another infinite sequence of zeros. Notice that the r.h.s. of (B.10) vanishes when $a = -b-2k,\ k \in \mathbb{Z}$. Show that those are also zeros of the l.h.s. (By symmetry between b and c, yet more zeros will be found.)

Lemma 4.6. *Let n be a nonnegative even integer, and b, c arbitrary, except nonnegative integers, and denote $d=(b+c-1)/2$. Then*

$$ {}_3F_2\left(\begin{matrix} -b-n, & b+n+1, & d \\ & b, \;\; c & \end{matrix}\right) = 0. \tag{4.11}$$

Proof. Through Karlsson's formula (B.12) and Gauss' formula (B.7) we transform (4.11) into the terminating ${}_3F_2$ with parameters $(-n-1,\, -b-n,\, d;\, b,\, -d-n)$, times a nonsingular factor. Such an ${}_3F_2$ falls under Dixon's formula [5, 3.1], [6, 4.4 (5)], the sum being a ratio of products of Γ functions. The only singular factor, $\Gamma(-n)$, appears in the denominator. (The factor $\Gamma((1-n)/2)$ is safe, because n is even.) Therefore, the ratio equals to 0. ■

§ 5. Proof of Theorem 4

The double integral $\tilde{L}'(m,n)$ will be expressed via a double sum using the Maclaurin expansions of the shifted Legendre polynomials. We rearrange the double sum and show that new partial single sums satisfy certain recurrence relation, from which the theorem follows.

We set aside the simple case $m=n=0$ and prove formulae (1.15) and (1.16) assuming $m = \max(m,n) > 0$.

5.1. Reduction of the integral $\tilde{B}'$ to a finite sum

Below *any function* means a function as good as a polynomial.

Denote the average value of a function f on an interval (a,b) by

$$\langle f\rangle_a^b = (b-a)^{-1}\int_a^b f(t)\,dt.$$

Introduce two operators, M and N, as follows:

$$Mf(x) = \langle f\rangle_0^x, \tag{5.1}$$

$$N f(x) = \int_0^x \langle f\rangle_t^x \, dt. \tag{5.2}$$

Clearly, if f is a polynomial of degree n then Mf is a polynomial of degree n, and Nf a polynomial of degree $n+1$.

Lemma 5.1. *Let f, g be any functions, and $G(x) = \int_0^x g(y)dy$. Then*

$$\int_0^1 f(x) \int_0^x \ln(x-y)\, g(y)\, dy\, dx = -\int_0^1 M[f\cdot G](x)\, dx - \int_0^1 f(x)\cdot Ng(x)\, dx. \tag{5.3}$$

Proof. Integration by parts w.r.t. y yields

$$\int_0^x \ln(x-y)\, g(y)\, dy \;=\; \ln x\; G(x) \;-\; Ng(x).$$

The assertion of Lemma follows by second integration

$$\int_0^1 \ln x\; f(x)\, G(x)\, dx \;=\; -\int_0^1 M[f\cdot G](x)\, dx,$$

where we remember that $\lim_{x\to 0} G(x) \ln x = 0$. □

Definition of the sums $S(m,n,r)$. Let $0 \le r \le (m+n)$. The quantities $S(m,n,r)$ are the coefficients of the generating function

$$p_m(x)\, q_{n+1}(x) \;=\; \sum_{r=0}^{m+n} S(m,n,r)\, x^{r+1}, \tag{5.4}$$

where q_{n+1} are the integrated shifted Legendre polynomials — see (A.12). The explicit formula is (cf. (A.11) and (A.12))

$$S(m,n,r) \;=\; (-1)^{m+n-r} \sum_{j=j\mathrm{min}}^{j\mathrm{max}} \frac{(m+j)!}{(m-j)!\, j!^2}\, \frac{(n+k)!}{(n-k)!\, k!\, (k+1)!}\,, \tag{5.5}$$

where $k = r-j$, $j\mathrm{min} = \max(0, r-n)$, $j\mathrm{max} = \min(m,r)$.
The promised double sum and its rearrangement are as follows

$$\begin{aligned} &\sum_{j=0}^{m}\sum_{k=0}^{n} \frac{(-1)^{m-j}\,(m+j)!}{(m-j)!\, j!^2}\, \frac{(-1)^{n-k}\,(n+k)!}{(n-k)!\, k!\, (k+1)!}\, \frac{1}{(j+k+2)^2} \\ &\qquad = \sum_{r=0}^{m+n} \frac{S(m,n,r)}{(r+2)^2} \;\stackrel{\text{def}}{=}\; K(m,n). \end{aligned} \tag{5.6}$$

Lemma 5.2. *For any m, n*

$$\iint\limits_{0<y<x<1} p_m(x)\, p_n(y)\, \ln(x-y)\, dy\, dx \;=\; -K(m,n) \;-\; \langle p_m,\, Np_n\rangle. \tag{5.7}$$

Proof. Evaluate the functional $u \to \langle Mu,\, 1\rangle$, which appears in (5.3), in terms of the Maclaurin expansion of the function $u(x)$

$$u(x) = \sum\nolimits_{r\geq 0} u_r\, x^r. \tag{5.8}$$

The operator M divides the r-th coefficient by $r+1$. Thus

$$\langle Mu,\, 1\rangle = \sum u_r\,(r+1)^{-2}.$$

Now take in (5.3) $f = p_m,\ g = p_n$, hence $u = p_m\, q_{n+1}$. Replacing u_{r+1} by $S(m,n,r)$, we arrive at (5.7). □

The scalar product $\langle p_m,\, Np_n\rangle$ can be found without difficulty, if $m \geq n$.

Lemma 5.3. (i) *If* $m \geq n+2$, *then* $\langle p_m,\, Np_n\rangle = 0$.
(ii) *Nonvanishing scalar products with* $m \geq n$ *are*

$$\langle p_n,\, Np_n\rangle = [(2n+1)(2n+2)]^{-1}, \tag{5.9}$$

$$\langle p_{n+1},\, Np_n\rangle = \frac{H_{n+1}}{2\,(2n+1)\,(2n+3)}, \tag{5.10}$$

where H_r *denotes the* r*-th harmonic sum*

$$H_r \;=\; 1 \,+\, 2^{-1} \,+ \ldots +\, r^{-1}. \tag{5.11}$$

Proof. (i) follows from orthogonality, since $\deg Np_n = n+1 < m$.
(ii) Apply the operator N to a Maclaurin series (5.8). If $U = \int u$, then

$$\langle u\rangle_t^x \;=\; \frac{U(x)-U(t)}{x-t} \;=\; \sum_{r\geq 0} \frac{u_r}{r+1} \sum_{j=0}^{r} x^{r-j}\, t^j.$$

Therefore,

$$Nu(x) = \sum_{r\geq 0} u_r\, \frac{h_{r+1}}{r+1}\, x^{r+1}. \tag{5.12}$$

Considering the expansion of Np_n, we obtain (in notation (A.11))

$$\begin{aligned} Np_n(x) \;&=\; a_n^{(n)}\, \frac{h_{n+1}}{n+1}\, x^{n+1} \,+\, a_{n-1}^{(n)}\, \frac{h_n}{n}\, x^n \\ &= \frac{a_n^{(n)}}{a_{n+1}^{(n+1)}}\, \frac{h_{n+1}}{n+1}\, p_{n+1}(x) \,+\, \left(\frac{a_{n-1}^{(n)}}{a_n^{(n)}}\, \frac{h_n}{n} \;-\; \frac{a_n^{(n+1)}}{a_{n+1}^{(n+1)}}\, \frac{h_{n+1}}{n+1} \right) p_n(x) + \ldots . \end{aligned}$$

Since $n^{-1}\, a_{n-1}^{(n)}/a_n^{(n)} \;=\; -1/2$ is independent of n, the coefficient at p_n in Np_n equals $(-1/2)\,(h_n - h_{n+1}) \;=\; (2n+2)^{-1}$. Recalling the value (A.8) of $\langle p_n, p_n\rangle$, we obtain (5.9). The coefficient at p_{n+1} in Np_n and consequently the value (5.10) also follow by (A.11) and (A.8). □

5.2. Recurrence for $S(m, n, r)$.

Lemma 5.4. *The following expression is symmetric in m and n $(m, n \geq 1)$*

$$D(m, n, r) \;=\; (n+1)\, S(m, n, r) \;-\; (n-1)\, S(m-1, n-1, r) \tag{5.13}$$

Proof. The generating function $D_{mn}(x)$ for the quantities $D(m, n, \cdot)$ is

$$\sum_{r=0}^{m+n} D(m, n, r)\, x^{r+1} \;=\; (n+1)\, p_m(x)\, q_{n+1}(x) \;-\; (n-1)\, p_{m-1}(x)\, q_n(x).$$

By (A.13),

$$D_{mn} \;=\; \frac{n+1}{2n+1}\, p_m\, (p_{n+1} - p_{n-1}) \;-\; \frac{n-1}{2n-1}\, p_{m-1}\, (p_n - p_{n-2}).$$

Apply (A.9) to p_{n+1} and p_n and find

$$D_{mn}(x) \;=\; \tilde{x}\, p_m\, p_n \;-\; p_m\, p_{n-1} \;-\; \frac{n-1}{n}\, \tilde{x}\, p_{m-1}\, p_{n-1} \;+\; \frac{n-1}{n}\, p_{m-1}\, p_{n-2}.$$

The first term is explicitly symmetric. Apply (A.9) to p_m in the 2nd term and obtain a symmetric formula for $D_{mn}(x) \;-\; \tilde{x}\, p_m\, p_n$:

$$\left(1 + \frac{n-1}{n} + \frac{m-1}{m}\right) \tilde{x}\, p_{m-1} p_{n-1} \;+\; \left(\frac{m-1}{m}\, p_{m-2} p_{n-1} + \frac{n-1}{n}\, p_{m-1} p_{n-2}\right).$$

□

Corollary. $K(m, n)$ satisfy the recurrence relation

$$\begin{aligned} &(n+1)\, K(m, n) \;-\; (m+1)\, K(n, m) \\ &\quad=\; (n-1)\, K(m-1, n-1) \;-\; (m-1)\, K(n-1, m-1). \end{aligned} \tag{5.14}$$

5.3. Completion of proof

Lemma 5.5. (i) *If $|m - n| \geq 2$ then $K(m, n) + K(n, m) = 0$.*
(ii) *In the cases when $|m - n|$, we have*

$$2\, K(n, n) \;=\; [n(2n+1)(2n+2)]^{-1}. \tag{5.15}$$

$$K(n+1, n) \;+\; K(n, n+1) \;=\; -[(2n+1)(2n+2)(2n+3)]^{-1}. \tag{5.16}$$

Proof. From proof of Lemma 5.2. we see that

$$K(m, n) + K(n, m) \;=\; \langle M[p_m\, q_{n+1} + q_{m+1}\, p_n],\, 1\rangle \;=\; \int_0^1 \frac{q_{m+1}(x)\, q_{n+1}(x)}{x}\, dx.$$

If $n < m-1$ then q_{m+1} is orthogonal to any polynomial of degree n (see (A.13)), in particular to $x^{-1} q_{n+1}(x)$. This proves (i).

(ii) Set

$$w_{n+1}(x) = \int_0^x x^{-1}\, q_{n+1}(x)\, dx.$$

Integrating by parts and remembering that $q_{n+1}(0) = q_{n+1}(1) = 0$ for $n > 0$, we get

$$K(m,n) + K(n,m) \;=\; -\int_0^1 p_m(x)\, w_{n+1}(x)\, dx.$$

Find the $(n+1)$-th and n-th Legendre coefficients of w_{n+1} by considering the respective Maclaurin coefficients. The calculation is very similar to that in proof of Lemma 5.3.

$$\begin{aligned} w_{n+1}(x) \;&=\; \frac{a_n^{(n)}}{(n+1)^2}\, x^{n+1} \;+\; \frac{a_{n-1}^{(n)}}{n^2}\, x^n \\ &=\; \frac{a_n^{(n)}}{a_{n+1}^{(n+1)}}\, \frac{p_{n+1}(x)}{(n+1)^2} \;+\; \left(\frac{a_{n-1}^{(n)}}{a_n^{(n)}}\, \frac{1}{n^2} \;-\; \frac{a_n^{(n+1)}}{a_{n+1}^{(n+1)}}\, \frac{1}{(n+1)^2} \right) p_n(x) + \ldots. \end{aligned}$$

The coefficient at p_n equals $(-1/2)\,[n^{-1} - (n+1)^{-1}] = -[2n(n+1)]^{-1}$. The coefficient at p_{n+1} is $[(2n+1)(2n+2)]^{-1}$. The proof is completed by using (A.8), as in Lemma 5.3. □

Lemma 5.6. *If* $m \geq 2$, *then*

$$K(m,0) \;=\; (-1)^{m+1}\, [m(m+2)(m^2-1)]^{-1}. \tag{5.17}$$

Proof. By (A.9),

$$2\, p_m(x)\, q_1(x) \equiv\; (\tilde{x}+1)\, p_m(x) = \frac{m+1}{2m+1}\, p_{m+1}(x) \;+\; p_m(x) \;+\; \frac{m}{2m+1}\, p_{m-1}(x).$$

Applying the operator M and integrating, we get

$$2\, K(m,0) \;=\; \int_0^1 \left(\frac{m+1}{2m+1}\, \frac{q_{m+2}(x)}{x} \;+\; \frac{q_{m+1}(x)}{x} \;+\; \frac{m}{2m+1}\, \frac{q_m(x)}{x} \right) dx.$$

Now use (A.14), and (5.17) follows . □

Final evaluation of $L'(m,n)$ requires separate treatment of the cases $m-n \geq 2$, $m = n$, and $m - n = 1$.

Case I: $m - n \geq 2$. In this case, $K(m,n) = -K(n,m)$ by Lemma 5.5. (i), and (5.14) becomes

$$(m+n+2)\, K(m,n) \;=\; (m+n-2)\, K(m-1,n-1).$$

The value $K(m,0)$ is known from (5.17). By induction we find

$$K(m,n) = \; -[(m+n)(m+n+2)((m-n)^2)]^{-1}.$$

Since the second term in the r.h.s. of (5.7) vanishes — see Lemma 5.3. (i) — we obtain the answer (1.15).

Case II: $m = n$. Using (5.15) and (5.9), we find the value of (5.7):

$$-\frac{1}{2n\,(2n+1)\,(2n+2)} - \frac{1}{(2n+1)\,(2n+2)} = \frac{1}{2n\,(2n+2)(-1)},$$

which agrees with (1.15).

Case III: $m = n+1$. Denote $K(n+1, n) = \lambda(n)$. The recurrence (5.14) with the nonhomogeneous symmetry relation (5.16) leads to the nonhomogeneous linear difference equation

$$(2n+3)\lambda_n - (2n-1)\lambda_{n-1} = \frac{2n+5}{(2n-1)\,(2n+1)_3}.$$

A solution of the homogeneous part is $\lambda_n^{(0)} = [(2n+1)(2n+3)]^{-1}$. Set $\lambda_n = \lambda_n^{(0)}\,\mu_n$. Then μ_n satisfies the equation

$$\mu_n - \mu_{n-1} = \frac{2n+5}{(2n-1)(2n+2)_2} = \frac{1}{2}\left(\frac{1}{2n-1} + \frac{1}{2n+3} - \frac{1}{n+1}\right).$$

By telescoping,

$$\mu_n = \mu_0 - \frac{1}{6} + H_{2n+1}^{\mathrm{odd}} - \frac{1}{2}H_{n+1} + \frac{1}{2}\left((2n+3)^{-1} - (2n+1)^{-1}\right). \tag{5.18}$$

Explicit calculation shows that $\mu_0 = 3K(1,0) = -1/12$. Substituting (5.18) together with (5.10) to (5.7), we obtain (1.16).

The proof is complete. ■

Excercise. Let m, n, l be nonnegative integers. Evaluate the sum:

$$T(m,n,l) = \sum_{p=0}^{m+n} \frac{S(m,n,p)}{p+l}.$$

Appendix A. Legendre polynomials $P_n(x)$

Definition. $P_n(x)$ is the polynomial of degree n such that: (i) $P_n(1) = 1$, and (ii) for any polynomial $f(x)$ of degree $< n$

$$\langle f,\, P_n\rangle \equiv \int_{-1}^{1} f(x)\,P_n(x)\,dx = 0.$$

Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}\,(x^2-1)^n. \tag{A.1}$$

Symmetry

$$P_n(-x) = (-1)^n P_n(x). \tag{A.2}$$

$\mathbf{L_2}$ norm

$$\langle P_n,\, P_n\rangle \;=\; 2\,(2n+1)^{-1}. \tag{A.3}$$

Recurrence relations

$$(n+1)\,P_{n+1}(x) \;=\; (2n+1)\,x\,P_n(x) \;-\; n\,P_{n-1}(x), \tag{A.4}$$

$$P'_{n+1}(x) \;=\; (2n+1)\,P_n(x) \;+\; P'_{n-1}(x). \tag{A.5}$$

Shifted Legendre polynomials [1, 22.2.11]

$$p_n(x) = P_n(\tilde{x}), \qquad \tilde{x} = 2x - 1, \tag{A.6}$$

form an orthogonal basis on $[0,1]$. When working with p_n's, we denote by $\langle\cdot,\cdot\rangle$ the integral scalar product on $[0,1]$. For reference, here are the shifted versions of the above formulae:

$$p_n(x) = (n!)^{-1}\,(d/dx)^n\,\left(\,x^n(x-1)^n\right)\,, \tag{A.7}$$

$$\langle p_n,\, p_n\rangle \;=\; (2n+1)^{-1}\,, \tag{A.8}$$

$$(n+1)\,p_{n+1}(x) \;=\; (2n+1)\,\tilde{x}\,p_n(x) \;-\; n\,p_{n-1}(x). \tag{A.9}$$

A bilinear identity for derivatives. If $r = |m-n| + 2s,\;\; s \geq 0$, then

$$\sum_{j+k=r+1} (-1)^j\, p_m^{(j)}(0)\, p_n^{(j)}(0) \;=\; 0. \tag{A.10}$$

Proof. The same combination of derivatives taken at 1 has opposite sign. So, by integration by parts, the l.h.s. equals to

$$(\pm 1/2)\left(\,\langle p_m,\, p_n^{(r+1)}\rangle \;-\; \langle p_m^{(r+1)},\, p_n\rangle\,\right),$$

which is 0, because, e.g., $\deg(p_m^{(r+1)}) = \max(0,\, n-1-2s) \;< n$. $\square$

Maclaurin expansion

$$p_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^k, \qquad a_k^{(n)} \;=\; \frac{(-1)^{n-k}\,(n+k)!}{(n-k)!\,k!^2}. \tag{A.11}$$

Integrated shifted Legendre polynomials $q_{n+1}(x)$

$$q_{n+1}(x) \;=\; \int_0^x p_n(t)\,dt \;=\; \sum_{k=0}^{n} \frac{a_k^{(n)}}{k+1}\, x^{k+1}. \tag{A.12}$$

These polynomials are used in Sect. 5. The notation is local to this work.

Decomposition (Cf. (A.5).) For $n > 0$,

$$2\,(2n+1)\,q_{n+1} \;=\; p_{n+1} \;-\; p_{n-1}. \tag{A.13}$$

Integral $\langle q_{n+1}, x^{-1}\rangle$. If $n > 0$, then

$$\int_0^1 \frac{q_{n+1}(x)}{x}\,dx \;\equiv\; -\int_0^1 p_n(x)\,\ln x\,dx \;=\; \frac{(-1)^n}{n(n+1)}. \tag{A.14}$$

Proof. Set $w(x) = x(x-1)$. By the Rodrigues formula $q_{n+1} = (n!)^{-1}\partial^{n-1}w^n$. Integrating by parts n times, we get

$$\int_{\varepsilon}^{1} \frac{q_{n+1}}{x}\,dx \;=\; \sum_{j=0}^{n-2} \frac{j!}{n!}\left[\frac{\partial^{n-2-j}\,w^n}{x^{j+1}}\right]_{\varepsilon}^{1} \;+\; \frac{(n-1)!}{n!}\int_{\varepsilon}^{1} \frac{w^n}{x^n}\,dx.$$

When $\varepsilon \to 0^+$, the only survivor is $n^{-1}\int_0^1 (x-1)^n\,dx = (-1)^n[n(n+1)]^{-1}$. □

Appendix B. Hypergeometric series and identities

Rising factorial (Pohhammer's symbol). Let $n \in \mathbb{Z}$, $a \in \mathbb{C}$. By definition, $(a)_n = \Gamma(a+n)/\Gamma(a)$. In particular, $(1)_n = n!$. Alternative definition:

$$\begin{array}{lll} \text{(i)} & (a)_0 \;=\; 1\,; & \\ \text{(ii)} & (a)_n \;=\; a(a+1)\cdot\ldots\cdot(a+n-1) & \text{if } n>0; \\ \text{(iii)} & (a)_n \;=\; [(a+n)_{-n}]^{-1} & \text{if } n<0. \end{array} \tag{B.1}$$

Three useful formulae:

$$(a)_{m+n} = (a)_m\,(a+m)_n, \tag{B.2}$$

$$(a)_n = (-1)^n\,(-a-n+1)_n, \tag{B.3}$$

$$(a)_n\,(a+1/2)_n = 4^{-n}\,(2a)_{2n}, \tag{B.4}$$

Hypergeometric series. The series depending on $p+q+1$ parameters

$${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,;\, z\right) \;=\; \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{n!\,(b_1)_n\ldots(b_q)_n}\,z^n. \tag{B.5}$$

is called the (generalized) hypergeometric series. The ordinary (Gauss') series is ${}_2F_1$. If one of the a_j's in (B.5) is a negative integer $-r$, then the series turns to a polynomial of degree r in z and is called **terminating**. In the frequent case $\mathbf{z=1}$, the argument z of ${}_pF_q$ is omitted.

Euler's integral for ${}_2F_1$ [1, (15.3.1)], [6, 2.1.3 (10)], [5, 1.5]**.**
If $\operatorname{Re} c > \operatorname{Re} b > 0$ and $z \notin [1,\infty)$, then

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\,;\, z\right) = \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}\,dt. \tag{B.6}$$

Gauss' formula for ${}_2F_1(\ldots;1)$ [1, (15.1.20)], [6, 2.1.3 (14)], [5, 1.3]

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\right) \;=\; \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (\operatorname{Re}(c-a-b)>0). \tag{B.7}$$

Thomae's transformation formula for $_3F_2$ [14, (2)], [5, 3.2], [8, (3.1.2)] If $s = e+f-a-b-c$, then

$$ {}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}\right) = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\ {}_3F_2\left(\begin{matrix} e-a, & f-a, & s \\ & s+b, & s+c \end{matrix}\right). \tag{B.8} $$

Whipple's summation formula for $_3F_2$ [14, (34)], [5, 3.4], [6, 4.4 (7)] . Terminating case: for a nonnegative integer n,

$$ {}_3F_2\left(\begin{matrix} -n, & n+1, & (b+c-1)/2 \\ & b, & c \end{matrix}\right) \;=\; 4^n\,\frac{(\frac{b-n}{2})_n\,(\frac{c-n}{2})_n}{(b)_n(c)_n}. \tag{B.9} $$

General case: let $a \in \mathbb{C}$, $a' = 1-a$, and $\mathrm{Re}\,(b+c) > 1$. Then

$$ {}_3F_2\left(\begin{matrix} a, & a', & (b+c-1)/2 \\ & b, & c \end{matrix}\right) \;=\; \frac{4\pi\,\cdot 2^{-(b+c)}\,\Gamma(b)\,\Gamma(c)}{\Gamma(\frac{b+a}{2})\,\Gamma(\frac{b+a'}{2})\,\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c+a'}{2}).} \tag{B.10} $$

Dougall's formula for $_5F_4$ [5, 4.4], [6, 4.5 (6)] . Suppose that

$$ 1+a \;=\; b+b' \;=\; c+c' \;=\; d+d' \;=\; e+e', $$

and $b-b'=1$. Then

$$ {}_5F_4\left(\begin{matrix} a, & b, & c, & d, & e \\ & b', & c', & d', & e' \end{matrix}\right) \;=\; \frac{\Gamma(c')\,\Gamma(d')\,\Gamma(e')\,\Gamma(1+a-c-d-e)}{\Gamma(1+a)\,\Gamma(c'-d)\,\Gamma(d'-e)\,\Gamma(e'-c)}. \tag{B.11} $$

Particular $_3F_2$ case of Karlsson's reduction formula [9]

$$ \begin{aligned} &{}_3F_2\left(\begin{matrix} -b-n, & b+n+1, & d \\ & b, & c \end{matrix}\right) \\ &\quad= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(-b-n)_k\,(d)_k}{(b)_k\,(c)_k}\ {}_2F_1\left(\begin{matrix} -b-n+k, & d+k \\ & c+k \end{matrix}\right). \end{aligned} \tag{B.12} $$

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, 1972.

[2] R. Askey, An integral of products of Legendre functions and a Clebsch-Gordan sum. Lett. Math. Phys. **6** (1982), no. 4, 299–302.

[3] R. Askey, E-mail posting on www.opsftalk.org, February 2001.

[4] R. Askey, G. Gasper, Inequalities for polynomials. The Bieberbach Conjecture, Proceedings of the Symposium on the Occasion of the Proof, ed. A. Baernstein II et al. Mathematical Surveys and Monographs, **21**, Amer. Math. Soc., Providence, 1986, 7–32.

[5] W.N.Bailey, Generalized Hypergeometric Series. Cambridge Univ. Press, Cambridge, 1935.

[6] Higher Transcendental Functions (Bateman Project), v.1. N.-Y: McGraw-Hill, 1953.

[7] A.M. Din, A simple sum formula for Clebsch-Gordan coefficients. Lett. Math. Phys. **5** (1981), no. 3, 207–211.

[8] G. Gasper, M. Rahman, Basic hypergeometric series. Encycl. of math. and its appl. **35**. Cambridge Univ. Press, Cambridge, 1990.

[9] P. Karlsson, Hypergeometric functions with integral parameter differences. J. Math. Phys. **12** (1971), 270–271.

[10] G.Polya, G.Szegö, Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen. J. Reine Angew. Math. **165** (1931), no 1, 4–49.

[11] M. Rahman, private communication. Feb. 1999.

[12] S. Yu. Sadov and K. A. McGreer, Galerkin method with Legendre bases for boundary integral equations in transmission problem with periodic piecewise-linear boundary. Direct and Inverse Problems in Electromagnetic Theory (Lviv, Ukraine, 1999), Proceedings, 55–58.

[13] M. A. Gilman, S. Yu. Sadov, A. S. Shamaev, S. I. Shamaev, Computer simulation of scattering of electromagnetic waves: some problems associated with remote radar sensing of sea surface. Radiotekhnika i Elektronika, Supplementary issue, **45** (2000), no. 2, 229–246.

[14] F.J.W. Whipple, A group of generalized hypergeometric series: relations between 120 allied series of the type $F[a, b, c; d, e]$. Proc. London Math.Soc. (2), **23** (1925), 104–114.